Stochastic resonance for nonlinear sensors with saturation

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We analyze the transmission of a noisy signal by sensor devices which are linear for small inputs and saturate at large inputs. Large information-carrying signals are thus distorted in their transmission. We demonstrate conditions where addition of noise to such large input signals can reduce the distortion that they undergo in the transmission. This is established for periodic, as well as aperiodic, and random information-carrying signals. Various measures characterizing the transmission, such as signal-to-noise ratio, input-output cross correlation, and mutual information, are shown improvable by addition of noise. These results constitute another instance of the nonlinear phenomenon of stochastic resonance where addition of noise enhances the signal.

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I. INTRODUCTION

The phenomenon of stochastic resonance establishes that, for certain types of nonlinear coupling between signal and noise, the presence or even the addition of noise, may result in improved performance for the signal [1,2]. Following its introduction some twenty years ago, stochastic resonance has gradually been observed in an increasing variety of nonlinear processes, including electronic circuits [3,4], optical devices [5,6], and neural systems [7,8]. It has also progressively been recognized that stochastic resonance can occur under many different forms, according to the nature of the signal, of the noise, of the nonlinear coupling which are involved and also of the measure of performance receiving improvement from the noise. Various forms of noise-enhanced transmission have been reported for periodic or aperiodic deterministic signals, or for random information-carrying signals, in the presence of Gaussian or non-Gaussian, white or colored, noise. Performance has been measured by signal-to-noise ratio, input-output gains, cross correlation, mutual information, channel capacity, detection probability, estimation efficacy, propagation distance, all these quantities have been shown improvable by addition of noise, in definite conditions. So far, systems that have been shown capable of producing a stochastic resonance effect essentially are nonlinear systems with potential barriers or with thresholds. In this case, the essence of the effect is that the information-carrying signal by itself is too small to overcome a threshold or a barrier in the response of the system. Addition of noise then allows some type of cooperation between signal and noise, so as to overcome the threshold or barrier, and elicit a response bearing stronger relation to the signal thanks to assistance from the noise.

In the present paper, we extend the class of nonlinear systems that have been shown capable of stochastic resonance. We consider static or memoryless systems which are purely linear in the small-signal limit (no threshold nor barrier). At the same time, the systems we consider exhibit saturation in their response for large input signals. Large information-carrying input signals are, thus, distorted in their transmission. We demonstrate conditions where addition of noise to such large input signals can reduce the distortion they undergo in the transmission, establishing another form of stochastic resonance.

II. A NONLINEAR TRANSMISSION

To have a simple demonstration of the stochastic resonance we envision, we consider an information signal $s(t)$ added to a white noise $\eta(t)$ endowed with a cumulative distribution function $F_\eta(u)$ and a probability density function $f_\eta(u)=dF_\eta(u)/du$. The input signal-plus-noise mixture $s(t)+\eta(t)$ is transmitted by a memoryless or static nonlinearity $g(.)$, so as to produce the output signal

$$y(t)=g[s(t)+\eta(t)]. \tag{1}$$

We shall consider here nonlinearities $g(.)$ which are linear for small inputs and saturate at large inputs. Together, we shall consider an information signal $s(t)$ of different types, successively deterministic periodic or aperiodic, or random. In each case, an appropriate measure of similarity between input $s(t)$ and output $y(t)$ will be introduced and investigated. We will show the possibility of increasing these measures of similarity through enhancement of the noise $\eta(t)$, thus, demonstrating stochastic resonance for each type of information signal with saturating nonlinearities.

III. PERIODIC SIGNAL

When $s(t)$ is deterministic periodic with period T_s, the output signal $y(t)$ of Eq. (1) generally is a cyclostationary random signal, with a power spectrum containing spectral lines at integer multiples of $1/T_s$ emerging out of a continuous noise background [9]. A standard measure of similarity of $y(t)$ with the T_s-periodic input $s(t)$ is a signal-to-noise ratio defined as the power contained in the output spectral line at $1/T_s$ divided by the power contained in the noise background in a small frequency band ΔB around $1/T_s$.

For the input-output relationship of Eq. (1), the power contained in the output spectral line at frequency n/T_s is given [9] by $|\overline{Y_n}|^2$, where $\overline{Y_n}$ is the order n Fourier coefficient of the T_s-periodic nonstationary output expectation $E[y(t)]$, i.e.,

$$\overline{Y_n} = \left\langle E[y(t)] \exp\left(-in\frac{2\pi}{T_s}t\right)\right\rangle, \tag{2}$$

with the time average defined as

$$\langle \cdots \rangle = \frac{1}{T_s}\int_0^{T_s} \cdots dt. \tag{3}$$

The output expectation $E[y(t)]$ at a fixed time t is computable as

$$E[y(t)] = \int_{-\infty}^{+\infty} g(u) f_\eta[u - s(t)] du. \tag{4}$$

The magnitude of the continuous noise background in the output spectrum is measured [9] by the stationarized output variance $\langle \mathrm{var}[y(t)]\rangle$, with the nonstationary variance $\mathrm{var}[y(t)] = E[y^2(t)] - E[y(t)]^2$ at a fixed time t, and

$$E[y^2(t)] = \int_{-\infty}^{+\infty} g^2(u) f_\eta[u - s(t)] du. \tag{5}$$

A signal-to-noise ratio $\mathcal{R}_n$, for the harmonic n/T_s in the ouput $y(t)$, follows as

$$\mathcal{R}_n = \frac{|\overline{Y_n}|^2}{\langle \mathrm{var}[y(t)]\rangle \Delta t \Delta B}, \tag{6}$$

where Δt is the time resolution of the measurement (i.e., the signal sampling period in a discrete-time implementation).

As a typical example of a saturating nonlinearity, we consider

$$g(u) = \tanh(\beta u) \tag{7}$$

with adjustable slope $\beta > 0$, which is linear as βu for small $|u| \ll 1/\beta$ and saturates at ± 1 for large $|u| \gg 1/\beta$.

Further, it is convenient for illustration to consider the case where $\eta(t)$ is a zero-mean uniform noise over $[-\sqrt{3}\sigma_\eta, \sqrt{3}\sigma_\eta]$ with standard deviation σ_η. In this case, with the nonlinearity $g(.)$ of Eq. (7), the integrals of Eqs. (4) and (5) can be evaluated analytically so as to yield

$$E[y(t)] = \frac{1}{2\sqrt{3}\beta\sigma_\eta} \ln\left[\frac{\cosh\{\beta[s(t) + \sqrt{3}\sigma_\eta]\}}{\cosh\{\beta[s(t) - \sqrt{3}\sigma_\eta]\}}\right] \tag{8}$$

and

$$E[y^2(t)] = \frac{1}{2\sqrt{3}\beta\sigma_\eta}[2\sqrt{3}\beta\sigma_\eta + \tanh\{\beta[s(t) - \sqrt{3}\sigma_\eta]\} - \tanh\{\beta[s(t) + \sqrt{3}\sigma_\eta]\}]. \tag{9}$$

Figure 1 shows the output signal-to-noise ratio $\mathcal{R}_1$ at frequency $1/T_s$ from Eq. (6), with $\Delta t \Delta B = 10^{-3}$, as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$, for the transmission of the periodic input $s(t) = 10 + 10\sin(2\pi t/T_s)$ by the nonlinearity of Eq. (7). Three values of the slope β are tested.

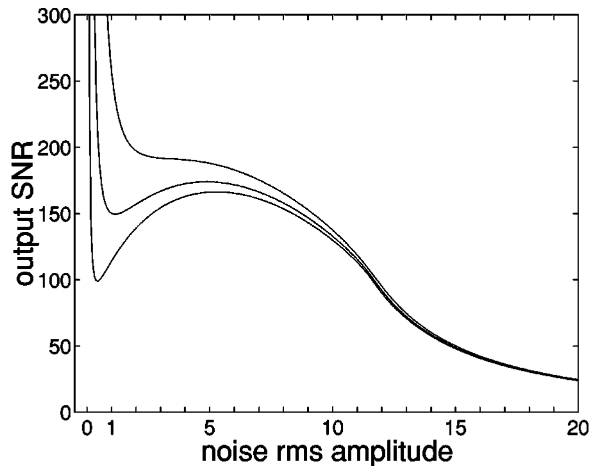

FIG. 1. Output signal-to-noise ratio $\mathcal{R}_1$ from Eq. (6) as a function of the rms amplitude σ_η of the uniform noise $\eta(t)$, for $s(t) = 10 + 10\sin(2\pi t/T_s)$ and in Eq. (7) with $\beta = 1$ (top), $\beta = 2$ (middle), $\beta = 5$ (bottom).

In the conditions of Fig. 1, the input $s(t) = 10 + 10\sin(2\pi t/T_s)$ displays excursions to large amplitudes, in relation to the parameter $1/\beta$ of the nonlinearity of Eq. (7). Therefore, $s(t)$ is strongly distorted in its transmission. In Fig. 1, at $\sigma_\eta \to 0$, the signal-to-noise ratio $\mathcal{R}_1$ gets infinite because, although the periodic component is very small in the output $y(t)$, the noise component vanishes. Next, as the noise level σ_η increases above zero, $\mathcal{R}_1$ rapidly drops. Yet, when σ_η becomes sufficiently large, $\mathcal{R}_1$ starts to rise. This is properly the stochastic resonance effect. The noise $\eta(t)$ added to the large input $s(t)$ makes it possible to operate the system in regions of the nonlinearity $\tanh[\beta(.)]$ that are more favorable to the transmission of $s(t)$. Thus, on average, the noise reduces the distortion experienced by the large input $s(t)$ in its transmission. This results in a signal-to-noise ratio $\mathcal{R}_1$ in Fig. 1 which can increase as σ_η is raised, to culminate for an optimal noise level where the distortion in the transmission of the periodic component is minimized. This effect of noise-assisted transmission is preserved when β is varied, and, as visible in Fig. 1, is more pronounced for large β when the distortion by the saturating nonlinearity is stronger.

IV. APERIODIC SIGNAL

When the deterministic input $s(t)$, we seek to extract out of the output $y(t)$, is no longer periodic, then the signal-to-noise ratio $\mathcal{R}_n$ of Eq. (6) is no longer available as a meaningful input-output measure of similarity. Consider $s(t)$ a deterministic aperiodic signal existing over the duration T_s. In such a case, meaningful input-output measures of similarity are provided by cross correlations as used for instance in Refs. [10,11]. We choose here to use the normalized time-averaged cross covariance between input $s(t)$ and output $y(t)$; it provides a similarity measure insensitive to both scaling and translation in signal amplitude. We introduce the signals centered around their time-averaged statistical expectation,

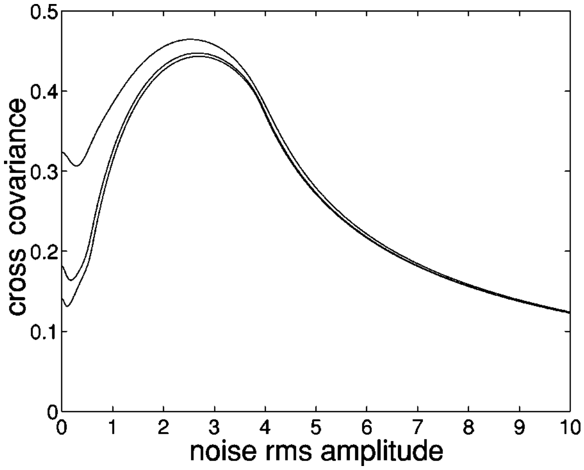

FIG. 2. Input-output normalized cross covariance C_{sy} from Eq. (13) as a function of the rms amplitude σ_η of the uniform noise $\eta(t)$, for $s(t)$ of Eq. (14) and in Eq. (7) with $\beta=2$ (top), $\beta=5$ (middle), $\beta=8$ (bottom).

$$\tilde{s}(t)=s(t)-\langle s(t)\rangle \tag{10}$$

and

$$\tilde{y}(t)=y(t)-\langle E[y(t)]\rangle, \tag{11}$$

with the time average again defined by Eq. (3). The normalized time-averaged cross covariance is

$$C_{sy}=\frac{\langle E[\tilde{s}(t)\tilde{y}(t)]\rangle}{\sqrt{\langle E[\tilde{s}^2(t)]\rangle\langle E[\tilde{y}^2(t)]\rangle}}, \tag{12}$$

or equivalently, since $s(t)$ is deterministic,

$$C_{sy}=\frac{\langle s(t)E[y(t)]\rangle-\langle s(t)\rangle\langle E[y(t)]\rangle}{\sqrt{[\langle s(t)^2\rangle-\langle s(t)\rangle^2][\langle E[y^2(t)]\rangle-\langle E[y(t)]\rangle^2]}}, \tag{13}$$

with $E[y(t)]$ and $E^2[y(t)]$ again given by Eqs. (4) and (5).

For an illustration of the possibility of stochastic resonance in the transmission of an aperiodic signal by a saturating sensor, we again consider the nonlinearity $g(.)$ of Eq. (7). Figure 2 shows the cross covariance from Eq. (13), as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$, for the transmission by the nonlinearity of Eq. (7) of the aperiodic input

$$s(t)=5\sin\left(2\pi\frac{t}{T_s/2}\right)+4\sin\left(2\pi\frac{t}{3T_s/2}\right), \tag{14}$$

when $t\in[0,T_s]$, and $s(t)=0$ elsewhere.

Again, Fig. 2 illustrates an effect of noise-assisted signal transmission, where the correlation between the aperiodic input $s(t)$ and the output $y(t)$ is maximized for a sufficient nonzero noise level. Figure 3 shows the large signal $s(t)$ of Eq. (14) and the way it is transmitted in the absence of noise and at the optimal noise level. Figure 3(c) displays an ensemble average of the output $y(t)$ showing that noise addition yields an output which is more similar to the input $s(t)$, on average, compared to the transmission with no noise of Fig. 3(b).

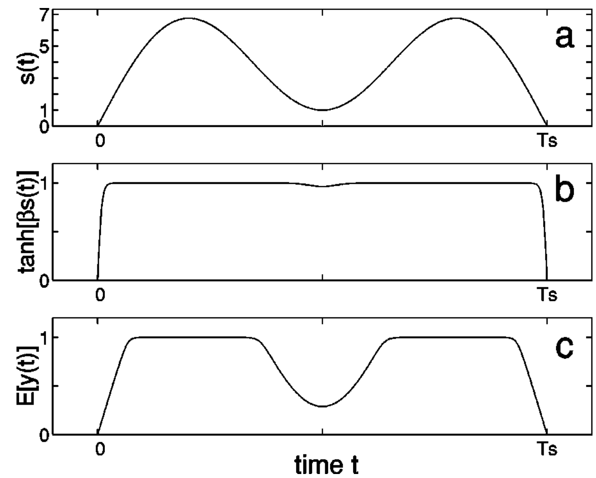

FIG. 3. Transmission by Eq. (7) with $\beta=2$. (a) Input signal $s(t)$ of Eq. (14). (b) Output $y(t)=\tanh[\beta s(t)]$ with no noise. (c) Ensemble average of output $y(t)=\tanh\{\beta[s(t)+\eta(t)]\}$ with $\eta(t)$ a zero-mean uniform noise at the optimum $\sigma_\eta=2.5$.

An alternative way can be used to quantify the benefit of adding noise. Figure 4 represents the ratio C_{sy}/C_{sx}, where $x(t)=s(t)+\eta(t)$ is the input signal-plus-noise mixture, and C_{sx} the normalized cross covariance between $s(t)$ and $x(t)$ computed as in Eqs. (12) and (13). The ratio C_{sy}/C_{sx} also can be increased by raising the noise, and it culminates at a maximum. The optimal value of the noise is different for C_{sy} of Fig. 2 and for C_{sy}/C_{sx} of Fig. 4, because these are two distinct quantitative measures of a qualitatively similar effect of noise-improved transmission.

V. RANDOM SIGNAL

For the transmission according to Eq. (1), we now consider the case where $s(t)$ is a random information-carrying

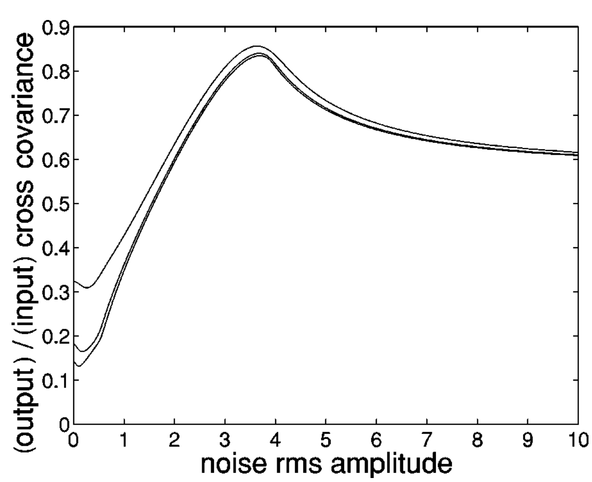

FIG. 4. Output/input ratio of the cross covariance C_{sy}/C_{sx} (see text) as a function of the rms amplitude σ_η of the uniform noise $\eta(t)$, for $s(t)$ of Eq. (14) and in Eq. (7) with $\beta=2$ (top), $\beta=5$ (middle), $\beta=8$ (bottom).

signal, chosen to be a white noise, statistically independent of $\eta(t)$. The mutual information $I(s;y)$ between input $s(t)$ and output $y(t)$ is defined as [12]

$$I(s;y)=\int\int p(s,y)\log\frac{p(s,y)}{p_s(s)p_y(y)}dsdy, \qquad (15)$$

where $p_s(s)$ and $p_y(y)$ are the marginal probability densities of $s(t)$ and $y(t)$, respectively, and $p(s,y)$ their joint probability density.

An approach to stochastic resonance would consist of studying $I(s;y)$ as a function of the level of the noise $\eta(t)$, and looking for conditions where $I(s;y)$ can be increased by raising $\eta(t)$. If the nonlinearity $g(.)$ of Eq. (1) is invertible, a complete transmission of information from $s(t)$ to $y(t)$ occurs at zero noise, and consequently no improvement of the input-output mutual information is obtainable by adding noise. We therefore turn, for a stochastic resonance effect, to a noninvertible transmission channel, under the form of a nonlinearity $g(.)$ very common for sensors with saturation, i.e.,

$$g(u)=\begin{cases}-1 & \text{for} \quad u\leqslant -1\\ u & \text{for} \quad -1<u<1\\ 1 & \text{for} \quad u\geqslant 1.\end{cases} \qquad (16)$$

In order to explicitly compute $I(s;y)$ in this case, we introduce a conditional probability density by $p(y,s)=p(y|s)\,p_s(s)$, which enables us to express Eq. (15) under the form

$$I(s;y)=\int ds p_s(s)\, I_1(s), \qquad (17)$$

where

$$I_1(s)=\int dy p(y|s)\log\frac{p(y|s)}{p_y(y)}. \qquad (18)$$

For any $y\in(-1,1)$, the linear part of $g(.)$ of Eq. (16), one has $p(y|s)=f_\eta(y-s)$. Also, given s, the probability that $y=-1$ is $\text{Prob}\{y=-1|s\}=\text{Prob}\{s+\eta<-1\}=F_\eta(-1-s)$. In a similar way, one has $\text{Prob}\{y=1|s\}=1-F_\eta(1-s)$. For any $y\in(-\infty,+\infty)$, the conditional density $p(y|s)$, through the use of the Dirac δ function $\delta(.)$, can, thus, be expressed as

$$p(y|s)=F_\eta(-1-s)\,\delta(y+1)+\overline{f}_\eta(y-s)+[1-F_\eta(1-s)]\,\delta(y-1), \qquad (19)$$

where $\overline{f}_\eta(y-s)$ coincides with $f_\eta(y-s)$ for $y\in(-1,1)$ and is zero for y elsewhere.

Next, we have

$$p_y(y)=\int p(y|s)p_s(s)ds. \qquad (20)$$

Therefore, thanks to Eqs. (19) and (20), the integral $I_1(s)$ of Eq. (18) is expressible as

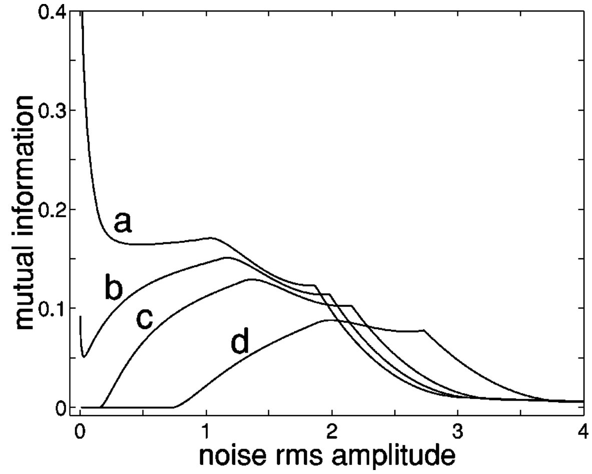

FIG. 5. Input-output mutual information $I(s;y)$ in shannon (log base 2) from Eq. (15), as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$, in the transmission by the saturating sensor of Eq. (16). The information signal $s(t)$ is uniform over $(-\sqrt{3}\sigma_s+m,\sqrt{3}\sigma_s+m)$ with $\sigma_s=1$ and $m=2.5$ (a), $m=2.7$ (b), $m=3$ (c), $m=4$ (d).

$$\begin{aligned}I_1(s)=&F_\eta(-1-s)\log\frac{F_\eta(-1-s)}{\int F_\eta(-1-s')p_s(s')ds'}\\&+\int_{-1}^{1}f_\eta(y-s)\log\frac{f_\eta(y-s)}{\int f_\eta(y-s')p_s(s')ds'}dy\\&+[1-F_\eta(1-s)]\\&\times\log\frac{1-F_\eta(1-s)}{\int[1-F_\eta(1-s')]p_s(s')ds'}.\end{aligned} \qquad (21)$$

Equation (17) together with Eq. (21) provide an explicit expression for the input-output mutual information $I(s;y)$, in the transmission by the saturating nonlinearity of Eq. (16), as a function of the statistical properties of the information signal $s(t)$ conveyed by $p_s(s)$ and of those of the noise $\eta(t)$ conveyed by $f_\eta(u)$ and $F_\eta(u)$. For some choices of $p_s(s)$ and $f_\eta(u)$, Eq. (21) and then Eq. (17) may be explicitly integrable in analytical forms; in other conditions, one may have to resort to numerical integrations. We shall now show the possibility of conditions, where the mutual information $I(s;y)$ can be increased by raising the level of the noise $\eta(t)$.

As a first example, we consider the case where $\eta(t)$ is a zero-mean noise uniform over $(-\sqrt{3}\sigma_\eta,\sqrt{3}\sigma_\eta)$, and $s(t)$ is uniform over $(-\sqrt{3}\sigma_s+m,\sqrt{3}\sigma_s+m)$ with mean $E[s(t)]=m$. In this case, Fig. 5 represents the input-output mutual information $I(s;y)$ from Eq. (15), in various conditions.

Figure 5 explicitly shows the possibility of conditions where $I(s;y)$ can be raised through an increase of the noise rms amplitude σ_η, over some ranges of σ_η. Especially,

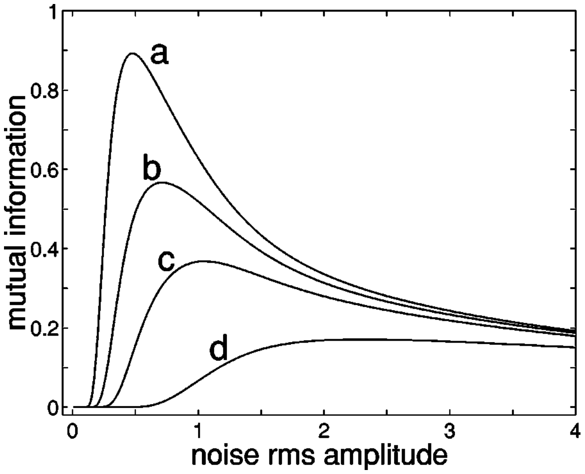

FIG. 6. Input-output mutual information $I(s;y)$ in shannon (log base 2) from Eq. (15), as a function of the rms amplitude σ_η of the zero-mean Gaussian noise $\eta(t)$, in the transmission by the saturating sensor of Eq. (16). The information signal $s(t)$ is a random bit stream with probability density $p_s(u)=[\delta(u+\sigma_s-m)+\delta(u-\sigma_s-m)]/2$ with $\sigma_s=1$ and $m=2.5$ (a), $m=2.7$ (b), $m=3$ (c), $m=4$ (d).

when the excursion of $s(t)$ is such that its lower bound $-\sqrt{3}\sigma_s+m$ is always above the saturation level $+1$ at the input of the sensor of Eq. (16), then the sensor output is always at saturation, and no information from $s(t)$ is transmitted onto $y(t)$ in the absence of noise. This is visible in Figs. 5(c),5(d), where $I(s;y)$ is zero when σ_η is zero. If noise then is added, a cooperative effect takes place between $s(t)$ and $\eta(t)$, whose result is to bring the sensor to operate, on some occasions, in its linear part, enabling a transmission of information between input $s(t)$ and output $y(t)$ with assistance from the noise $\eta(t)$. This favorable outcome is maximized by a sufficient nonzero optimal amount of noise, as visible in Fig. 5. This is a form of stochastic resonance, or noise-aided transmission of information through saturating sensors. A similar stochastic resonance effect is also possible when the signal lower bound $-\sqrt{3}\sigma_s+m$ is not always above the saturation level $+1$ at zero noise. This is the case in Figs. 5(a),5(b), where at zero noise $s(t)$ evolves partly below the saturation in the linear region of the sensor, whence a nonzero $I(s;y)$ when σ_η is zero. Then, a nonmonotonic influence of the added noise follows when σ_η is raised above zero. A degradation of $I(s;y)$ appears first at low σ_η, but which is followed by a possibility of enhancing $I(s;y)$ by further raising σ_η, over some ranges of σ_η.

As another example, we consider the case where $\eta(t)$ is a zero-mean Gaussian noise, and $s(t)$ is a random binary signal with probability density $p_s(u)=[\delta(u+\sigma_s-m)+\delta(u-\sigma_s-m)]/2$. This $s(t)$ describes a random stream of two equiprobable binary symbols fluctuating at $\pm\sigma_s$ away from the mean level m. In this case, Fig. 6 represents the input-output mutual information $I(s;y)$ from Eq. (15), in various conditions.

In the conditions of Fig. 6, the bit stream $s(t)$ fluctuates always above the saturation level $+1$ of the sensor of Eq. (16), thus, strictly no information is transmitted at $\sigma_\eta=0$ in the absence of noise. As soon as σ_η is raised above zero, transmission of information becomes possible in principle, because of the infinite wings of the Gaussian density, but is very inefficient at small σ_η. It is only for a sufficiently large noise level σ_η that a substantial amount of information transmission takes place, associated to a maximum of $I(s;y)$ in Fig. 6. This again represents a form of stochastic resonance or noise-aided information transmission.

VI. CONCLUSION

We have shown that the transmission of an information-carrying signal by sensors linear for small inputs and saturating for large inputs, can be improved by addition of noise. This was established for periodic as well as aperiodic and for random information-carrying signals. Various measures characterizing the transmission, including signal-to-noise ratio, input-output cross correlation and mutual information, were shown improvable by addition of noise. The conditions we have considered here to exhibit stochastic resonance in saturating nonlinearities are merely illustrative, and the effect is preserved in many other conditions. Together with its quantitative assessment, a qualitative explanation of the effect is that for large signals, not well positioned in relation to the saturating nonlinearity, addition of noise at the input acts as a random bias shifting, on average, the operating zone of the sensor towards its linear part, more favorable to the transmission of the signal. These results can be interpreted as an instance of the general phenomenon of stochastic resonance, by which nonlinear transmission of a signal can be improved by addition of noise. Such an effect can be useful when a signal has to be transmitted by nonlinear systems over which no full control is available, especially to adjust the operating zone of the nonlinearity in accordance with the signal.

A class of natural systems, achieving very efficient signal processing, and in which stochastic resonance has been shown to operate, is formed by neural systems. In neuronal transmission, stochastic resonance has been reported essentially in the region of the neuron threshold, to assist small subthreshold signals. Yet, in addition to their threshold, neurons also exhibit saturation in their response. In the region of the neuron saturation, the form of stochastic resonance we have reported here could also play a role for improved performance. In a speculative way, this could happen for instance in the perception of visual information in the presence of a very high level of ambiant light bringing the system close to saturation, or in the hearing of acoustic signals in the presence of suddenly high aerostatic pressure.

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